Explicitly symmetrical treatment of three-body phase space

2004 J. Phys. A: Math. Gen. 37 4871

(http://iopscience.iop.org/0305-4470/37/17/016)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4871–4886 PII: S0305-4470(04)73959-1

Explicitly symmetrical treatment of three-body phase space

A I Davydychev[1,2,3,4] **and R Delbourgo**[1]

[1] School of Mathematics and Physics, University of Tasmania, GPO Box 252-21, Hobart, Tasmania 7001, Australia
[2] Institute for Nuclear Physics, Moscow State University, 119992 Moscow, Russia
[3] Department of Physics, University of Mainz, Staudingerweg 7, D-55099 Mainz, Germany

E-mail: davyd@thep.physik.uni-mainz.de and bob.delbourgo@utas.edu.au

Received 17 December 2003, in final form 16 March 2004
Published 14 April 2004
Online at stacks.iop.org/JPhysA/37/4871 (DOI: 10.1088/0305-4470/37/17/016)

Abstract
We derive expressions for three-body phase space that are *explicitly* symmetrical in the masses of the three particles. We study geometrical properties of the variables involved in elliptic integrals and demonstrate that it is convenient to use the Jacobian zeta function to express the results in four and six dimensions.

PACS numbers: 11.80.Cr, 11.10.Kk, 12.30.Gp

1. Introduction

The subject of three-body (and, in general, N-body) relativistic phase space is as old as the hills and one might well think that all that there is to know is already known. In numerical and experimental terms this is indeed true: for a long time Dalitz plots [1, 2] have been routinely used in picturing data and they prove extremely helpful for picking out resonant intermediate states of particular spin by their preferential population of the plots. In the absence of any amplitude modulation by resonances or otherwise, the plots are at their blandest as they just represent three-body phase space.

In any multibody production such as $A + B \to 1 + 2 + \cdots + N$, the probability of the process is largely governed by the total momentum $p = p_A + p_B$, the masses of the final particles $m_1, m_2, \ldots, m_N$ relative to $\sqrt{p^2}$ and an overall coupling constant. Surely there is also the dynamics of production which modulates the coupling magnitude by intermediate state contributions, but the overall rate is mainly influenced by the unmodulated phase-space integral as written below. The case of $N = 3$ phase-space integrals and the manifest symmetry of the result upon the three masses of the product particles is the subject of this paper.

[4] Present address: Schlumberger, SPC, 155 Industrial Dr, Sugar Land, TX 77478, USA.

One of the first comprehensive references on this subject is the paper by Almgren [3]. In his normalization, the integral over the N-particle phase space is defined as

$$I_N^{(D)}(p, m_1, \ldots, m_N) = \int \cdots \int \left\{ \prod_{i=1}^{N} \mathrm{d}^D p_i \delta\big(p_i^2 - m_i^2\big)\theta\big(p_i^0\big) \right\} \delta\left(\sum_{i=1}^{N} p_i - p \right) \tag{1.1}$$

where p is the total momentum. From now on, we will frequently use the notation $p = \sqrt{p^2}$, since usually it is easy to distinguish it from the cases when the four-dimensional vector p is meant. As a rule, we will also omit the arguments of $I_N^{(D)}$. In four dimensions ($D = 4$), we will denote $I_N \equiv I_N^{(4)}$ (this is the original Almgren's notation). More details about integrals in other dimensions can be found in [4] and in the rest of this paper. It is worth mentioning that $I_N^{(D)}$ is easy to work out for odd values of D, whereas considering even values of D brings in elliptic functions and is more difficult.

For kinematical reasons, it is clear that the results for the integrals (1.1) have no physical meaning if the absolute value of the momentum p is less than the sum of the masses. Therefore, in what follows we will imply that all results for $I_N^{(D)}$ are accompanied by $\theta\{p^2 - (m_1 + \cdots + m_N)^2\}$, without writing this theta function explicitly. In [5, 3], integral recurrence relations for I_N (at $D = 4$) were discussed. For an arbitrary dimension D, the generating relation can be presented as

$$I_N^{(D)}(p, m_1, \ldots, m_N) = \int \mathrm{d}s\, I_{R+1}^{(D)}(p, \sqrt{s}, m_{N-R+1}, \ldots, m_N) I_{N-R}^{(D)}(\sqrt{s}, m_1, \ldots, m_{N-R}). \tag{1.2}$$

Taking into account the theta functions associated with $I_{N-R}^{(D)}$ and $I_{R+1}^{(D)}$, one can see that the actual limits of the integration variable s in equation (1.2) extend from $\big(\sum_{i=1}^{N-R} m_i\big)^2$ to $\big(p - \sum_{i=N-R+1}^{N} m_i\big)^2$. Once we fix the subsets of masses in the arguments of the integrals on the r.h.s. of (1.2), the explicit symmetry gets lost. It is clear, however, that equation (1.2) still contains that symmetry, since one can split the masses $m_1, \ldots, m_N$ into these two subsets in any possible way. Another type of integral recurrence relations for $I_N^{(D)}$, with respect to the value of D, was considered in [6].

The simplest example is the two-particle phase space, $N = 2$. In this case, the phase-space integral (1.1) in four dimensions can be easily evaluated as

$$I_2 = \frac{\pi}{2p^2} \sqrt{\lambda\big(p^2, m_1^2, m_2^2\big)} \tag{1.3}$$

where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \tag{1.4}$$

is nothing but the well-known Källen function [7].

Using equations (1.2) and (1.3) (for the case $D = 4$, $R = 1$), one can obtain the following integral representation [3, 8] for the three-particle ($N = 3$) phase space:

$$I_3 = \frac{\pi^2}{4p^2} \int_{s_2}^{s_3} \frac{\mathrm{d}s}{s} \sqrt{(s - s_1)(s - s_2)(s_3 - s)(s_4 - s)} \tag{1.5}$$

with

$$s_1 = (m_1 - m_2)^2 \qquad s_2 = (m_1 + m_2)^2 \qquad s_3 = (p - m_3)^2 \qquad s_4 = (p + m_3)^2 \tag{1.6}$$

so that $s_1 \leqslant s_2 \leqslant s_3 \leqslant s_4$. The result of the calculation of the integral (1.5) can be expressed in terms of the elliptic integrals [3, 8] (for convenience, we collect the definitions and relevant properties of elliptic integrals in the appendix),

$$I_3 = \frac{\pi^2}{4p^2\sqrt{Q_+}}\left\{\frac{1}{2}Q_+\left(m_1^2+m_2^2+m_3^2+p^2\right)E(k)\right.$$
$$+4m_1m_2[(p-m_3)^2-(m_1-m_2)^2][(p+m_3)^2-m_3p+m_1m_2]K(k)$$
$$+8m_1m_2\left[\left(m_1^2+m_2^2\right)\left(p^2+m_3^2\right)-2m_1^2m_2^2-2m_3^2p^2\right]\Pi\left(\alpha_1^2,k\right)$$
$$\left.-8m_1m_2\left(p^2-m_3^2\right)^2\Pi\left(\alpha_2^2,k\right)\right\} \tag{1.7}$$

where we use the following notations:

$$Q_+ \equiv (p+m_1+m_2+m_3)(p+m_1-m_2-m_3)(p-m_1+m_2-m_3)(p-m_1-m_2+m_3)$$
$$Q_- \equiv (p-m_1-m_2-m_3)(p-m_1+m_2+m_3)(p+m_1-m_2+m_3)(p+m_1+m_2-m_3) \tag{1.8}$$

$$k \equiv \sqrt{\frac{Q_-}{Q_+}} \qquad \alpha_1^2 = \frac{(p-m_3)^2-(m_1+m_2)^2}{(p-m_3)^2-(m_1-m_2)^2} \qquad \alpha_2^2 = \frac{(m_1-m_2)^2}{(m_1+m_2)^2}\alpha_1^2. \tag{1.9}$$

We note that in [8] the notation $q_{\pm\pm} \equiv (p \pm m_3)^2 - (m_1 \pm m_2)^2$ was used. In particular, we have $Q_+ = q_{++}q_{--}$, $Q_- = q_{+-}q_{-+}$, $k^2 = q_{+-}q_{-+}/(q_{++}q_{--})$, $\alpha_1^2 = q_{-+}/q_{--}$. Note that $Q_\pm$ differ by the sign of p only.

It is clear from definition (1.1) that I_3 should be a symmetrical function of the three masses m_1, m_2, m_3. The representation (1.7) in terms of elliptic integrals is however *not* explicitly symmetrical in the masses, although it must be implicitly so. One may, of course, generate a symmetrical form by averaging the unsymmetrical-looking expressions over the three possible permutations of m_i, but this would be 'cheating' since each of them should be symmetrical by itself, although this is hardly transparent.

Note that the quantities Q_+ and Q_- (and, therefore, the argument k) are totally symmetric in m_1, m_2, m_3. (In fact, they are symmetric in all four arguments p, m_1, m_2, m_3.) Therefore, the term containing $E(k)$ in equation (1.7) is also symmetric. The function $K(k)$ itself is also symmetric, but its coefficient is not symmetric. We also note that the product of Q_+ and Q_- produces the quantity

$$D_{123} \equiv Q_+Q_- = [p^2-(m_1+m_2+m_3)^2][p^2-(-m_1+m_2+m_3)^2]$$
$$\times [p^2-(m_1-m_2+m_3)^2][p^2-(m_1+m_2-m_3)^2] \tag{1.10}$$

that occurs in recurrence relations for the sunset diagram (see, e.g., in [9, 10]). It should be noted that the imaginary part of the sunset diagram is proportional to the three-particle phase-space integral. For instance, in the notation of [8], $\mathrm{Im}(T_{123}) = -4\pi^{-1}I_3$. We also note that ρ_N^D considered in [4, 6] are related to $I_N^{(D)}$ as $\rho_N^D = (2\pi)^{N+D-ND}I_N^{(D)}$.

For equal masses, $m_1 = m_2 = m_3 \equiv m$, equation (1.7) yields

$$\frac{\pi^2}{4p^2}\sqrt{(p-m)(p+3m)}\left\{\frac{1}{2}(p-m)(p^2+3m^2)E(k_{\mathrm{eq}})-4m^2pK(k_{\mathrm{eq}})\right\} \tag{1.11}$$

with

$$k_{\mathrm{eq}} = \sqrt{\frac{(p+m)^3(p-3m)}{(p-m)^3(p+3m)}}. \tag{1.12}$$

Some other special cases of equation (1.7) are described in [8, 11].

This paper is devoted to a new way of exhibiting the results in an *explicitly* symmetrical manner. To do this, we will employ another integral representation for I_3, in terms of

Mandelstam variables s, t, u [12] and the Kibble cubic $\Phi(s,t,u)$ [13]. In particular, we will show that it is convenient to present the result (1.7) in terms of Jacobi Z function whose definition is given in the appendix.

2. Phase-space integrals

As an illustration, let us demonstrate how the connection with the Dalitz figure can be derived directly from definition (1.1). The D-dimensional vector p can be presented as $(p^0, \mathbf{p})$, where $\mathbf{p}$ is the $(D-1)$-dimensional Euclidean vector of space components. Without loss of generality, we can work in the centre-of-mass frame, $p = (p^0, \mathbf{0})$. Using the integral representation

$$\delta\left(\sum_{i=1}^{N} p_i - p\right) = \frac{1}{(2\pi)^D}\int \mathrm{d}^D x\ \exp\left\{\mathrm{i}\sum_{i=1}^{N}(p_i x) - \mathrm{i}(px)\right\} \tag{2.1}$$

with $(px) = p^0 x^0$, we get

$$I_N^{(D)} = \frac{1}{(2\pi)^D}\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}p^0x^0}\left\{\prod_{i=1}^{N}\int \mathrm{d}^D p_i \delta\big(p_i^2 - m_i^2\big)\theta\big(p_i^0\big)\,\mathrm{e}^{\mathrm{i}(p_i x)}\right\}. \tag{2.2}$$

(Similar method was used in [14].) Integrating over $(D-1)$-dimensional angles of $\mathbf{p}_i$ we get

$$\int \mathrm{d}^D p_i \delta\big(p_i^2 - m_i^2\big)\,\theta\big(p_i^0\big)\,\mathrm{e}^{\mathrm{i}(p_i x)} = \frac{(2\pi)^{(D-1)/2}}{2\xi^{(D-3)/2}}\int_0^\infty \frac{\rho_i^{(D-1)/2}\mathrm{d}\rho_i}{\sqrt{\rho_i^2+m_i^2}} J_{(D-3)/2}(\rho_i\xi)\,\mathrm{e}^{\mathrm{i}x^0\sqrt{\rho_i^2+m_i^2}} \tag{2.3}$$

with $\rho_i \equiv |\mathbf{p}_i|$ and $\xi \equiv |\mathbf{x}|$. In the four-dimensional case the Bessel function reduces to an elementary function, $J_{1/2}(\rho_i\xi) = [2/(\pi\rho_i\xi)]^{1/2}\sin(\rho_i\xi)$. We note an analogy with the calculation of Feynman integrals in the coordinate space [15, 22], when each massive propagator yields a (modified) Bessel function.

Let us consider, for example, the two-particle phase space. Then, the integration over ξ gives us

$$\int_0^\infty \xi\mathrm{d}\xi\ J_\nu(\rho_1\xi)J_\nu(\rho_2\xi) = 2\delta\big(\rho_1^2 - \rho_2^2\big)$$

with $\nu = (D-3)/2$, so that we can put $\rho_1 = \rho_2 \equiv \rho$, whereas the integration over x^0 yields another delta function, $\delta\big(p - \sqrt{\rho^2+m_1^2} - \sqrt{\rho^2+m_2^2}\big)$ in the centre-of-mass frame. The resulting integral

$$I_2^{(D)} = \frac{\pi^{(D-1)/2}}{2\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty \frac{\rho^{D-2}\mathrm{d}\rho}{\sqrt{\rho^2+m_1^2}\sqrt{\rho^2+m_2^2}}\,\delta\left(p - \sqrt{\rho^2+m_1^2} - \sqrt{\rho^2+m_2^2}\right) \tag{2.4}$$

can be easily evaluated, yielding (see, e.g., in [6])

$$I_2^{(D)} = \frac{\pi^{(D-1)/2}}{(2p)^{D-2}\Gamma\left(\frac{D-1}{2}\right)}\left[\lambda\big(p^2, m_1^2, m_2^2\big)\right]^{(D-3)/2} \tag{2.5}$$

where λ is defined in equation (1.4). For $D = 4$, equation (2.5) reduces to the well-known answer (1.3).

For the three-particle phase-space integral we get

$$I_3^{(D)} = \frac{2^{(D-7)/2}\pi^{D-2}}{\Gamma\left(\frac{D-1}{2}\right)}\int_0^\infty \frac{\mathrm{d}\xi}{\xi^{(D-5)/2}}\int_{-\infty}^{\infty}\mathrm{d}x^0\,\mathrm{e}^{-\mathrm{i}p^0x^0}$$
$$\times\prod_{i=1}^{3}\int_0^\infty \frac{\rho_i^{(D-1)/2}\mathrm{d}\rho_i}{\sqrt{\rho_i^2+m_i^2}} J_{(D-3)/2}(\rho_i\xi)\,\mathrm{e}^{\mathrm{i}x^0\sqrt{\rho_i^2+m_i^2}}. \tag{2.6}$$

Here we can integrate over ξ, using (see [16])

$$\int_0^\infty \frac{\mathrm{d}\xi}{\xi^{\nu-1}} J_\nu(\rho_1\xi) J_\nu(\rho_2\xi) J_\nu(\rho_3\xi) = \frac{2\theta\{-\lambda(\rho_1^2, \rho_2^2, \rho_3^2)\}[-\lambda(\rho_1^2, \rho_2^2, \rho_3^2)]^{\nu-1/2}}{\pi^{1/2}\Gamma\left(\nu+\frac{1}{2}\right)(8\rho_1\rho_2\rho_3)^\nu} \tag{2.7}$$

(with $\nu = (D-3)/2$), where λ is the Källen function (1.4). In fact, in our case, when all $\rho_i \geqslant 0$,

$$\theta\{-\lambda(\rho_1^2, \rho_2^2, \rho_3^2)\} = \theta(\rho_1 + \rho_2 - \rho_3)\theta(\rho_2 + \rho_3 - \rho_1)\theta(\rho_3 + \rho_1 - \rho_2), \tag{2.8}$$

i.e. it equals 1 when one can compose a triangle with sides ρ_1, ρ_2, ρ_3, and gives 0 otherwise (cf equation (11) of [17]).

Introducing notation $\sigma_i = \sqrt{\rho_i^2 + m_i^2}$ and integrating over x^0 (getting a δ function) we arrive at

$$I_3^{(D)} = \frac{\pi^{D-2}}{\Gamma(D-2)} \int_{m_1}^\infty \int_{m_2}^\infty \int_{m_3}^\infty \mathrm{d}\sigma_1\, \mathrm{d}\sigma_2\, \mathrm{d}\sigma_3 \delta(p - \sigma_1 - \sigma_2 - \sigma_3) \\ \times \left[-\lambda(\sigma_1^2 - m_1^2, \sigma_2^2 - m_2^2, \sigma_3^2 - m_3^2)\right]^{(D-4)/2} \theta\left\{-\lambda(\sigma_1^2 - m_1^2, \sigma_2^2 - m_2^2, \sigma_3^2 - m_3^2)\right\}. \tag{2.9}$$

In four dimensions the factor $[-\lambda]^{(D-4)/2}$ disappears and, geometrically, we need to calculate a closed area on the plane $\sigma_1 + \sigma_2 + \sigma_3 = p^0 \equiv p$, with the boundary of the figure described by

$$\lambda(\sigma_1^2 - m_1^2, \sigma_2^2 - m_2^2, \sigma_3^2 - m_3^2) = 0 \qquad \sigma_1 + \sigma_2 + \sigma_3 = p. \tag{2.10}$$

Furthermore, introducing Mandelstam-type variables

$$s = p^2 + m_3^2 - 2p\sigma_3 \qquad t = p^2 + m_1^2 - 2p\sigma_1 \qquad u = p^2 + m_2^2 - 2p\sigma_2 \tag{2.11}$$

satisfying

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + p^2 \equiv w_0 \tag{2.12}$$

one arrives at another integral representation (the limits of integration are discussed below),

$$I_3^{(D)} = \frac{\pi^{D-2}}{4p^{D-2}\Gamma(D-2)} \iiint \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}u\, \delta(s + t + u - w_0) [\Phi(s, t, u)]^{(D-4)/2} \theta\{\Phi(s, t, u)\} \tag{2.13}$$

where

$$\Phi(s, t, u) = -\frac{1}{16p^2}\lambda\left\{\lambda(s, m_3^2, p^2), \lambda(t, m_1^2, p^2), \lambda(u, m_2^2, p^2)\right\} \tag{2.14}$$

can also be written in a more familiar Kibble cubic form [13],

$$\Phi(s, t, u) = stu - s(m_1^2 m_2^2 + p^2 m_3^2) - t(m_2^2 m_3^2 + p^2 m_1^2) - u(m_3^2 m_1^2 + p^2 m_2^2) \\ + 2(m_1^2 m_2^2 m_3^2 + p^2 m_1^2 m_2^2 + p^2 m_2^2 m_3^2 + p^2 m_3^2 m_1^2) \tag{2.15}$$

provided that the condition (2.12) is satisfied. In particular, in four dimensions we have

$$I_3 = \frac{\pi^2}{4p^2} \iiint \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}u\, \delta(s + t + u - w_0)\theta\{\Phi(s, t, u)\}. \tag{2.16}$$

According to definition (2.11) in terms of σ_i, one can see that the maximal values of s, t and u (corresponding to the upper limits of integration in equations (2.13) and (2.16)) are $s_{\max} = (p - m_3)^2$, $t_{\max} = (p - m_1)^2$ and $u_{\max} = (p - m_2)^2$. To define the minimal values of s, t and u, the familiar Dalitz–Kibble plot given in figure 1 is useful. Due to

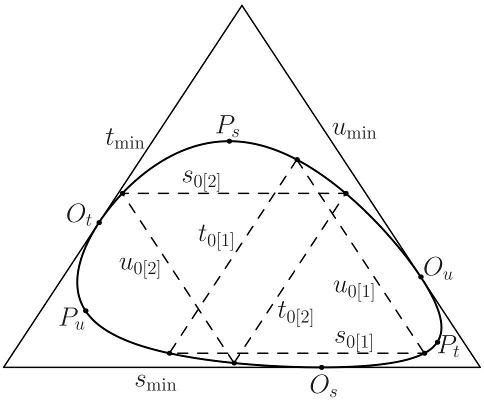

Figure 1. The Dalitz–Kibble integration area.

the condition (2.12), the region of integration is restricted by a triangle, with $s \geqslant s_{\min} = (m_1 + m_2)^2$, $t \geqslant t_{\min} = (m_2 + m_3)^2$ and $u \geqslant u_{\min} = (m_1 + m_3)^2$. Moreover, due to the theta function $\theta\{\Phi(s, t, u)\}$ the region of integration is in fact restricted by the interior of the cubic curve $\Phi(s, t, u) = 0$, see figure 1. Within that area, the Mandelstam variables s, t and u take their minimal values in points O_s, O_t and O_u, respectively, whereas their maximal values correspond to the points P_s, P_t and P_u. (The dashed triangles will be discussed in section 4.)

The function $\Phi(s, t, u)$ has a maximum within the region of integration. For equal masses, the maximal value $\Phi_{\max} = \frac{1}{27} p^2 (p^2 - 9m^2)^2$ occurs at $s = t = u = \frac{1}{3}(p^2 + 3m^2)$. For the general unequal masses, one needs to solve a fourth-order algebraic equation to find the position of the maximum.

We note that the representation (2.14) can be extracted from equation (5.39) of [18], using symmetry properties. Our $\Phi(s, t, u)$ corresponds to $-G\big(s, t, p^2, m_2^2, m_1^2, m_3^2\big)$, in the notations of [18]. The G-function is symmetric with respect to the permutations of three pairs of arguments, (s, t), $\big(p^2, m_2^2\big)$ and $\big(m_1^2, m_3^2\big)$. Although the authors presume from their equation (5.39) that 'from a practical point of view this identity is not very useful', we found that its symmetric form is certainly helpful in understanding the structural properties of phase-space integrals.

3. Geometrical interpretation

Let us introduce

$$c_{12} = \frac{s - m_1^2 - m_2^2}{2m_1 m_2} \qquad c_{23} = \frac{t - m_2^2 - m_3^2}{2m_2 m_3} \qquad c_{13} = \frac{u - m_1^2 - m_3^2}{2m_1 m_3}. \tag{3.1}$$

Then, the function $\Phi(s, t, u)$ can be presented as a Gram determinant,

$$\Phi(s, t, u) = 4m_1^2 m_2^2 m_3^2 \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix} \tag{3.2}$$

whereas the δ function becomes

$$\delta\big(s + t + u - m_1^2 - m_2^2 - m_3^2 - p^2\big)$$
$$\Rightarrow \delta\left(m_1^2 + m_2^2 + m_3^2 + 2m_1 m_2 c_{12} + 2m_2 m_3 c_{23} + 2m_1 m_3 c_{13} - p^2\right). \tag{3.3}$$

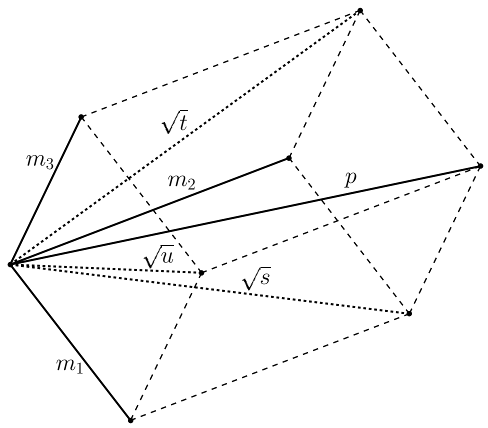

Figure 2. The parallelepiped interpretation.

In this way, we get

$$I_3 = \frac{2\pi^2}{p^2} m_1^2 m_2^2 m_3^2 \iiint \mathrm{d}c_{12}\, \mathrm{d}c_{13}\, \mathrm{d}c_{23} \theta \left(\begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix} \right) \times \delta \left(m_1^2 + m_2^2 + m_3^2 + 2m_1 m_2 c_{12} + 2m_2 m_3 c_{23} + 2m_1 m_3 c_{13} - p^2 \right) \tag{3.4}$$

where the integration extends over $c_{jl} \geqslant 1$.

If one were to interpret c_{jl} as the cosines of the angles between the m_j and m_l sides of a vertex of a parallelepiped (formed by m_1, m_2 and m_3, see figure 2), then all these quantities would have a straightforward geometrical interpretation. Namely, $\Phi(s, t, u)$ would be 4{volume of parallelepiped}^{2}, whereas the δ function would tell us that the 'principal' diagonal of this parallelepiped should be equal to p. In this case, the quantities $\sqrt{s}$, $\sqrt{t}$ and $\sqrt{u}$ could be identified as the diagonals of the faces of the parallelepiped, see figure 2. Moreover,

$$\frac{p^2 + m_1^2 - t}{2pm_1} \qquad \frac{p^2 + m_2^2 - u}{2pm_2} \qquad \text{and} \qquad \frac{p^2 + m_3^2 - s}{2pm_3} \tag{3.5}$$

could be understood as cosines of the angles between the diagonal p and the m_i sides of the parallelepiped. In other words, these are the angles between p and m_i in triangles with sides $(p, m_1, \sqrt{t})$, $(p, m_2, \sqrt{u})$ and $(p, m_3, \sqrt{s})$, respectively.

Using this geometrical figure, we can mention a rather interesting geometrical meaning of equation (2.14). Namely, it tells us that the volume of the parallelepiped is $(8/p)$ times the area of triangle whose sides are given by the areas of triangles formed out of the principal diagonal p, one of the face diagonals ($\sqrt{s}$, $\sqrt{t}$ or $\sqrt{u}$), and the appropriate m_3, m_1 or m_2 side.

However, when we are above the threshold, $p^2 > (m_1+m_2+m_3)^2$, the quantities c_{jl} exceed one and therefore the expressions should be understood in the sense of analytic continuation, i.e. as hyperbolic cosines. The same is valid for the triangles $(p, m_3, \sqrt{s})$, etc: they should also be understood in the sense of analytic continuation, since $p \geqslant m_3 + \sqrt{s}$, etc. Therefore, the quantities σ_i/m_i should also be understood as hyperbolic cosines, whereas $\sqrt{\sigma_i^2 - m_i^2}\big/m_i$ are hyperbolic sines.

Nevertheless, in the region below the threshold (which we need, for instance, to describe the real part of the sunset diagram), this geometrical figure can have direct meaning, generalizing the figure we had for the one-loop two-point function [19].

4. Kibble cubic characteristics

Suppose

$$(s_0, t_0, w_0 - s_0 - t_0) \qquad (s_0, w_0 - s_0 - u_0, u_0) \qquad (w_0 - t_0 - u_0, t_0, u_0) \tag{4.1}$$

all are the roots of the equation $\Phi(s, t, u) = 0$. Then, we can present $\Phi(s, t, u)$ as

$$\Phi(s, t, u) = stu - st_0u_0 - s_0tu_0 - s_0t_0u + 2s_0t_0u_0. \tag{4.2}$$

Furthermore, if we shift the Mandelstam variables as

$$s = s_0 + s' \qquad t = t_0 + t' \qquad u = u_0 + u' \tag{4.3}$$

subject to the condition

$$s' + t' + u' = w_0 - s_0 - t_0 - u_0 \equiv w_0' \tag{4.4}$$

then

$$\Phi(s, t, u) \Rightarrow s't'u' + s_0t'u' + s't_0u' + s't'u_0. \tag{4.5}$$

Using equation (4.2) and defining

$$c_{tu} \equiv \sqrt{\frac{t_0u_0}{tu}} \qquad c_{st} \equiv \sqrt{\frac{s_0t_0}{st}} \qquad c_{su} \equiv \sqrt{\frac{s_0u_0}{su}} \tag{4.6}$$

we arrive at another Gram determinant representation for $\Phi(s, t, u)$ (cf equation (3.2)),

$$\Phi(s, t, u) = stu \begin{vmatrix} 1 & c_{tu} & c_{st} \\ c_{tu} & 1 & c_{su} \\ c_{st} & c_{su} & 1 \end{vmatrix}. \tag{4.7}$$

There are (at least) two sets of solutions (4.1) that can be described as

$$s_0 = \frac{A_1A_2}{A_3} \qquad t_0 = \frac{A_2A_3}{A_1} \qquad u_0 = \frac{A_1A_3}{A_2} \tag{4.8}$$

so that equation (4.2) yields

$$\Phi(s, t, u) = stu - A_1^2t - A_2^2u - A_3^2s + 2A_1A_2A_3. \tag{4.9}$$

The first set of solutions corresponds to

$$A_1 \equiv pm_1 + m_2m_3 \qquad A_2 \equiv pm_2 + m_3m_1 \qquad A_3 \equiv pm_3 + m_1m_2. \tag{4.10}$$

For this set, we have

$$w_0' = w_0 - s_0 - t_0 - u_0 = \frac{m_1m_2m_3pQ_-}{A_1A_2A_3} \tag{4.11}$$

$$c_{tu} = \frac{pm_3 + m_1m_2}{\sqrt{tu}} \qquad c_{st} = \frac{pm_2 + m_1m_3}{\sqrt{st}} \qquad c_{su} = \frac{pm_1 + m_2m_3}{\sqrt{su}}. \tag{4.12}$$

Note that if we change $p \to -p$ in equation (4.10), this would also be a solution, which would correspond to a 'non-physical' branch of the Kibble cubic.

The second set of solutions corresponds to

$$\begin{aligned} A_1 &\equiv \tfrac{1}{2}\left(p^2 + m_1^2 - m_2^2 - m_3^2\right) \qquad A_2 \equiv \tfrac{1}{2}\left(p^2 - m_1^2 + m_2^2 - m_3^2\right) \\ A_3 &\equiv \tfrac{1}{2}\left(p^2 - m_1^2 - m_2^2 + m_3^2\right). \end{aligned} \tag{4.13}$$

For this set, we get

$$w_0' = w_0 - s_0 - t_0 - u_0 = -\frac{Q_+Q_-}{16A_1A_2A_3} = -\frac{D_{123}}{16A_1A_2A_3} \tag{4.14}$$

$$c_{tu} = \frac{p^2 - m_1^2 - m_2^2 + m_3^2}{2\sqrt{tu}} \qquad c_{st} = \frac{p^2 - m_1^2 + m_2^2 - m_3^2}{2\sqrt{st}} \qquad c_{su} = \frac{p^2 + m_1^2 - m_2^2 - m_3^2}{2\sqrt{su}}. \tag{4.15}$$

It should be noted that the value of s_0 corresponding to the second set satisfies

$$\lambda\big(s_0, m_1^2, m_2^2\big) = \lambda\big(s_0, p^2, m_3^2\big) \tag{4.16}$$

i.e. the areas (or their analytical continuations) of triangles with sides $(\sqrt{s_0}, m_1, m_2)$ and $(\sqrt{s_0}, p, m_3)$ are equal. Analogously,

$$\lambda\big(t_0, m_2^2, m_3^2\big) = \lambda\big(t_0, p^2, m_1^2\big) \qquad \lambda\big(u_0, m_1^2, m_3^2\big) = \lambda\big(u_0, p^2, m_2^2\big). \tag{4.17}$$

Moreover, one can get the direct geometrical interpretation of the quantities (4.15) through the familiar parallelepiped shown in figure 2. Namely, c_{tu} is nothing but the cosine between the face diagonals $\sqrt{t}$ and $\sqrt{u}$. Accordingly, c_{su} is the cosine of the angle between $\sqrt{s}$ and $\sqrt{u}$ diagonals, whilst c_{st} is the cosine of the angle between $\sqrt{s}$ and $\sqrt{t}$ diagonals. If we construct a tetrahedron using the $\sqrt{s}$, $\sqrt{t}$ and $\sqrt{u}$ diagonals then, according to equation (4.7), $\Phi(s, t, u)$ would represent 36 times its volume squared.

In the Dalitz–Kibble plot shown in figure 1 we connect the points (4.1) for each of the two sets by dashed lines, introducing subscripts $_{[1]}$ and $_{[2]}$ for the first and the second set, respectively. The two resulting 'dashed' triangles indicate that the two sets of solutions are complementary to each other. Namely, the boundary of the Dalitz plot confines the products tu, st and su as follows:

$$\begin{aligned}
(pm_3 + m_1m_2)^2 &\leqslant tu \leqslant \tfrac{1}{4}\big(p^2 - m_1^2 - m_2^2 + m_3^2\big)^2 \\
(pm_2 + m_1m_3)^2 &\leqslant st \leqslant \tfrac{1}{4}\big(p^2 - m_1^2 + m_2^2 - m_3^2\big)^2 \\
(pm_1 + m_2m_3)^2 &\leqslant su \leqslant \tfrac{1}{4}\big(p^2 + m_1^2 - m_2^2 - m_3^2\big)^2
\end{aligned} \tag{4.18}$$

or, equivalently,

$$\begin{aligned}
&(t_0u_0)_{[1]} \leqslant tu \leqslant (t_0u_0)_{[2]} \qquad (s_0t_0)_{[1]} \leqslant st \leqslant (s_0t_0)_{[2]} \\
&(s_0u_0)_{[1]} \leqslant su \leqslant (s_0u_0)_{[2]}.
\end{aligned} \tag{4.19}$$

In other words, the first and the second sets yield, respectively, the minimal and the maximal values of tu, st and su.

Let us consider the corresponding values of the 'cosines' c_{su}, c_{st} and c_{tu}. For the first set, c_{su}, c_{st} and c_{tu} would vary between 1 and $\cos\varphi_i$ $(i = 1, 2, 3)$, respectively, where

$$\begin{aligned}
&\cos\varphi_1 = \frac{2(pm_1 - m_2m_3)}{p^2 + m_1^2 - m_2^2 - m_3^2} \qquad \cos\varphi_2 = \frac{2(pm_2 - m_3m_1)}{p^2 - m_1^2 + m_2^2 - m_3^2} \\
&\cos\varphi_3 = \frac{2(pm_3 - m_1m_2)}{p^2 - m_1^2 - m_2^2 + m_3^2}.
\end{aligned} \tag{4.20}$$

For the second set, c_{su}, c_{st} and c_{tu} would vary between 1 and $1/\cos\varphi_i$. This means that we need to understand them in the sense of analytic continuation.

The angles φ_i will be very important below. Their sines can be presented as

$$\begin{aligned}
&\sin\varphi_1 = \frac{\sqrt{Q_+}}{p^2 + m_1^2 - m_2^2 - m_3^2} \qquad \sin\varphi_2 = \frac{\sqrt{Q_+}}{p^2 - m_1^2 + m_2^2 - m_3^2} \\
&\sin\varphi_3 = \frac{\sqrt{Q_+}}{p^2 - m_1^2 - m_2^2 + m_3^2}.
\end{aligned} \tag{4.21}$$

It is interesting that the corresponding Gram determinant can be factorized as

$$\begin{vmatrix} 1 & -\cos\varphi_3 & -\cos\varphi_2 \\ -\cos\varphi_3 & 1 & -\cos\varphi_1 \\ -\cos\varphi_2 & -\cos\varphi_1 & 1 \end{vmatrix} = \frac{1}{k^2}\sin^2\varphi_1\sin^2\varphi_2\sin^2\varphi_3. \qquad (4.22)$$

Equation (4.22) can be used to express k in terms of φ_i. We also note that

$$\tan\frac{\varphi_3}{2} = \sqrt{\frac{(p-m_3)^2-(m_1-m_2)^2}{(p+m_3)^2-(m_1+m_2)^2}} \qquad (4.23)$$

and similarly for φ_1 and φ_2. In particular, one can see that at the threshold, $p = m_1 + m_2 + m_3$, the angles φ_i are related to the angles θ_i from equation (20) of [20] (see also [10]) as $\varphi_i = \pi - 2\theta_i$, and

$$(\varphi_1+\varphi_2+\varphi_3)|_{p=m_1+m_2+m_3} = \pi. \qquad (4.24)$$

We can also consider associated angles ψ_i, such that

$$\sin\psi_i = k\sin\varphi_i \qquad \cos\psi_i = \sqrt{1-k^2\sin^2\varphi_i}. \qquad (4.25)$$

Explicitly, we get

$$\sin\psi_3 = \frac{\sqrt{Q_-}}{p^2-m_1^2-m_2^2+m_3^2} \qquad \cos\psi_3 = \frac{2(pm_3+m_1m_2)}{p^2-m_1^2-m_2^2+m_3^2} \qquad (4.26)$$

etc. For these angles, we get

$$\begin{vmatrix} 1 & \cos\psi_3 & \cos\psi_2 \\ \cos\psi_3 & 1 & \cos\psi_1 \\ \cos\psi_2 & \cos\psi_1 & 1 \end{vmatrix} = \frac{1}{k^2}\sin^2\psi_1\sin^2\psi_2\sin^2\psi_3. \qquad (4.27)$$

5. A naturally symmetric representation

Using the representation (4.2) for $\Phi(s,t,u)$, in terms of s_0, t_0 and u_0, the three-body phase-space integral can be written as

$$I_3 = \frac{\pi^2}{4p^2}\iiint \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\,\delta(s+t+u-w_0)\theta(stu-st_0u_0-s_0tu_0-s_0t_0u+2s_0t_0u_0) \qquad (5.1)$$

with $w_0 = p^2+m_1^2+m_2^2+m_3^2$. Integrating over u yields

$$I_3 = \frac{\pi^2}{4p^2}\iint \mathrm{d}s\,\mathrm{d}t\theta\{(st-s_0t_0)(w_0-s-t)-st_0u_0-s_0tu_0+2s_0t_0u_0\}. \qquad (5.2)$$

Then, integrating over t, we basically obtain the difference between the roots of the quadratic argument of the θ function, which is

$$\frac{1}{s}\sqrt{s^4-2w_0s^3+\left(w_0^2+2s_0t_0+2s_0u_0-4t_0u_0\right)s^2-2(w_0t_0+w_0u_0-4u_0t_0)s_0s+s_0^2(t_0-u_0)^2}.$$

It is easy to check that for both sets of (s_0, t_0, u_0) the square root takes the familiar form (1.5), which yields the non-symmetric result (1.7) in terms of elliptic integrals.

Starting from the representation (2.13), one can easily generalize the result (1.5) to the D-dimensional case as

$$I_3^{(D)} = \frac{\pi^{D-1}}{(4p)^{D-2}\Gamma^2\left(\frac{D-1}{2}\right)}\int_{s_2}^{s_3}\frac{\mathrm{d}s}{s^{D/2-1}}[(s-s_1)(s-s_2)(s_3-s)(s_4-s)]^{(D-3)/2} \qquad (5.3)$$

with s_i given in equation (1.6). Another way to derive the representation (5.3) is to use the recurrence relation (1.2),

$$I_3^{(D)} = \int_{s_2}^{s_3} \mathrm{d}s\, I_2^{(D)}(p, \sqrt{s}, m_3) I_2^{(D)}(\sqrt{s}, m_1, m_2) \tag{5.4}$$

and substitute the result (2.5) for $I_2^{(D)}$. The result (5.3) corresponds to equation (9) of [4]. (We note that the overall factor on the r.h.s. of equation (9) of [4] should be corrected: $(32\pi)^{2-2\ell}$ should be changed into $\frac{1}{2}(16\pi)^{2-2\ell}$, with $\ell = D/2$).

Using representation (5.3), it is easy to see (just substituting $s = x^2$) that all *odd*-dimensional phase-space integrals can be expressed in terms of polynomial functions (see, e.g., in [21–23]),

$$I_3^{(3)} = \frac{\pi^2}{2p}(p - m_1 - m_2 - m_3) \tag{5.5}$$

$$\begin{aligned} I_3^{(5)} = \frac{\pi^4}{60p^3}(p - m_1 - m_2 - m_3)^3 \bigg[&\frac{1}{7}(p - m_1 - m_2 - m_3)^4 + (m_1 + m_2 + m_3)p^3 \\ &- 2\big(m_1^2 + m_2^2 + m_3^2\big)p^2 + \big(m_1^3 + m_2^3 + m_3^3\big)p + 12 m_1 m_2 m_3 p \\ &- (m_1 + m_2 + m_3)(m_1 + m_2)(m_2 + m_3)(m_3 + m_1) \\ &+ 4 m_1 m_2 m_3 (m_1 + m_2 + m_3) \bigg] \end{aligned} \tag{5.6}$$

etc, which are explicitly symmetric in the masses m_i. However, the results in *even* dimensions appear to be less trivial.

It is instructive to consider the two-dimensional case, $D = 2$. Then, the integral (5.3) yields just the elliptic integral $K(k)$,

$$I_3^{(2)} = \int_{s_2}^{s_3} \frac{\mathrm{d}s}{\sqrt{(s - s_1)(s - s_2)(s_3 - s)(s_4 - s)}} = \frac{2}{\sqrt{Q_+}} K(k). \tag{5.7}$$

This is of course explicitly symmetric in the masses without further ado. On the other hand, using the δ function in equation (5.1), we can insert $1 = (s + t + u)/w_0$ in the integrand, and then consider the three resulting terms (with s, t and u) separately. In this way, we arrive at an alternative expression,

$$\begin{aligned} I_3^{(2)} = \frac{1}{w_0} \Bigg\{ &\int_{s_2}^{s_3} \frac{s\, \mathrm{d}s}{\sqrt{(s - s_1)(s - s_2)(s_3 - s)(s_4 - s)}} + \int_{t_2}^{t_3} \frac{t\, \mathrm{d}t}{\sqrt{(t - t_1)(t - t_2)(t_3 - t)(t_4 - t)}} \\ &+ \int_{u_2}^{u_3} \frac{u\, \mathrm{d}u}{\sqrt{(u - u_1)(u - u_2)(u_3 - u)(u_4 - u)}} \Bigg\} \end{aligned} \tag{5.8}$$

where the roots t_i and u_i can be obtained from s_i given in equation (1.6) by proper permutation of the masses m_i. Each of the integrals involved in equation (5.8) can be expressed in terms of Jacobian Z function (see the appendix). For example,

$$\int_{s_2}^{s_3} \frac{s\, \mathrm{d}s}{\sqrt{(s - s_1)(s - s_2)(s_3 - s)(s_4 - s)}} = \frac{\sin\varphi_3}{\sin\varphi_1 \sin\varphi_2} K(k) - K(k) Z(\varphi_3, k) \tag{5.9}$$

where φ_i are nothing but the three angles defined in equations (4.20) and (4.21). Comparing the resulting expression with the original result (5.7), we obtain a very useful relation between the three $Z(\varphi_i, k)$ functions,

$$Z(\varphi_1, k) + Z(\varphi_2, k) + Z(\varphi_3, k) = k^2 \sin\varphi_1 \sin\varphi_2 \sin\varphi_3. \tag{5.10}$$

Let us now consider the four-dimensional integral $I_3^{(4)} \equiv I_3$, namely, its representation (1.5). A useful observation is that the result would be simpler if we managed to get rid of s in the denominator. In particular, it would contain just one elliptic integral Π, rather than two. How are we to eliminate s in the denominator? Again, using the δ function in equation (5.1), we can insert $1 = (s+t+u)/w_0$ in the integrand, and then consider the three resulting terms (with s, t and u) separately. For the term with s, we perform the t and u integrations and arrive at the same integral as in equation (1.5), but without s in the denominator. In two other integrals, we just integrate in a different order, leaving as the last one the t or u integration, respectively. In this way, we obtain for the integral (5.1)

$$\frac{\pi^2}{4p^2 w_0}\int\!\!\int\!\!\int \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u (s+t+u)\delta(s+t+u-w_0)\theta(stu - st_0u_0 - s_0tu_0 - s_0t_0u + 2s_0t_0u_0)$$

$$= \frac{\pi^2}{4p^2 w_0}\left\{\int_{s_2}^{s_3} \mathrm{d}s\sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)} + \int_{t_2}^{t_3} \mathrm{d}t\sqrt{(t-t_1)(t-t_2)(t_3-t)(t_4-t)} + \int_{u_2}^{u_3} \mathrm{d}u\sqrt{(u-u_1)(u-u_2)(u_3-u)(u_4-u)}\right\} \tag{5.11}$$

where, as before, the roots t_i and u_i can be obtained from s_i given in equation (1.6) by permutation of the masses.

Using the formulae given in [24] along with equations (A.8) and (A.11), the s-integral in equation (5.11) can be calculated in terms of a Jacobian Z function (see the appendix),

$$\int_{s_2}^{s_3} \mathrm{d}s\sqrt{(s-s_1)(s-s_2)(s_3-s)(s_4-s)} = \sqrt{Q_+}\left\{2\left(p^2m_3^2 - m_1^2m_2^2\right)K(k)\frac{Z(\varphi_3,k)}{\sin\varphi_3} + \frac{1}{6}Q_- K(k) + \frac{1}{6}\left[\left(p^2 - m_1^2 - m_2^2 + m_3^2\right)^2 + 8\left(p^2m_3^2 + m_1^2m_2^2\right)\right][E(k) - K(k)]\right\} \tag{5.12}$$

where φ_3 is one of the three angles defined in equations (4.20) and (4.21).

Collecting the results for all three integrals and using the relation (5.10), we arrive at the symmetric result

$$I_3 = \frac{\pi^2}{8p^2}\left\{\sqrt{Q_+}\left(p^2 + m_1^2 + m_2^2 + m_3^2\right)[E(k) - K(k)] + Q_+ K(k)\left[\frac{Z(\varphi_1,k)}{\sin^2\varphi_1} + \frac{Z(\varphi_2,k)}{\sin^2\varphi_2} + \frac{Z(\varphi_3,k)}{\sin^2\varphi_3}\right]\right\}. \tag{5.13}$$

This symmetric result can also be presented in terms of the elliptic integrals Π, using (see in [24])

$$K(k)Z(\varphi_i,k) = \cot\varphi_i\sqrt{1-k^2\sin^2\varphi_i}\,[\Pi(k^2\sin^2\varphi_i,k) - K(k)]. \tag{5.14}$$

In principle, one can also derive the result (5.13) directly from the non-symmetric representation (1.7) (see [25]), in a tedious way relying on the use of several relations collected in the appendix, including the addition formula (A.9) for Jacobi Z functions.

It is worth noting that in a similar way one can obtain results for higher even dimensions D. For instance, in six dimensions we get

$$I_3^{(6)} = \frac{\pi^4}{144p^4}\left\{\frac{Q_+^{1/2}}{20}[E(k)-K(k)]\Big[192\big(p^8+m_1^8+m_2^8+m_3^8\big)-112\big(p^4+m_1^4+m_2^4+m_3^4\big)^2\right.$$
$$-6\big(p^2+m_1^2+m_2^2+m_3^2\big)^4-156\big(p^6+m_1^6+m_2^6+m_3^6\big)\big(p^2+m_1^2+m_2^2+m_3^2\big)$$
$$+83\big(p^4+m_1^4+m_2^4+m_3^4\big)\big(p^2+m_1^2+m_2^2+m_3^2\big)^2\Big]$$
$$+\frac{1}{40}Q_-Q_+^{1/2}K(k)\big[3\big(p^2+m_1^2+m_2^2+m_3^2\big)^2-16\big(p^4+m_1^4+m_2^4+m_3^4\big)\big]$$
$$+\frac{3}{4}\frac{Q_+^{5/2}K(k)}{\sin\varphi_1\sin\varphi_2\sin\varphi_3}\left[\frac{Z(\varphi_1,k)}{\sin^2\varphi_1}+\frac{Z(\varphi_2,k)}{\sin^2\varphi_2}+\frac{Z(\varphi_3,k)}{\sin^2\varphi_3}\right]$$
$$\left.-\frac{3}{8}Q_+^2\big(p^2+m_1^2+m_2^2+m_3^2\big)K(k)\left[\frac{Z(\varphi_1,k)}{\sin^4\varphi_1}+\frac{Z(\varphi_2,k)}{\sin^4\varphi_2}+\frac{Z(\varphi_3,k)}{\sin^4\varphi_3}\right]\right\}. \tag{5.15}$$

As an alternative way to obtain results for higher values of D, the approach of the paper [9] may be used.

In the equal-mass case,

$$\varphi_1=\varphi_2=\varphi_3\equiv\varphi_{\rm eq} \qquad \sin\varphi_{\rm eq}=\frac{\sqrt{(p-m)(p+3m)}}{p+m} \qquad \cos\varphi_{\rm eq}=\frac{2m}{p+m}. \tag{5.16}$$

Here, using equation (5.10) we get

$$Z(\varphi_{\rm eq},k_{\rm eq})=\tfrac{1}{3}k_{\rm eq}^2\sin^3\varphi_{\rm eq} \tag{5.17}$$

with $k_{\rm eq}$ defined in equation (1.12). In this way, we reproduce equation (1.11), whereas for $D=6$ equation (5.15) yields

$$\frac{\pi^4}{2880p^4}\sqrt{(p-m)^3(p+3m)}\big\{(p^4-9m^4)(p^4-42p^2m^2+9m^4)[E(k_{\rm eq})-K(k_{\rm eq})]$$
$$+(p+m)^3(p-3m)(p^4-36p^2m^2+27m^4)K(k_{\rm eq})\big\}. \tag{5.18}$$

We note that equation (5.17) yields a reduction formula of $Z(\varphi,k)$, for a special case when

$$k=\frac{\sqrt{1-2\cos\varphi}}{\sin\varphi(1-\cos\varphi)}.$$

Another interesting limit corresponds to the case when one of the masses vanishes (for example, $m_3\to 0$). This corresponds to the case $k\to 1$, when $E(k)$ is finite ($E(1)=1$) whereas $K(k)$ develops logarithmic singularity. At $m_3=0$, $\cos\varphi_3<0$ and $\varphi_3>\pi/2$, so that we need to use equation (A.7). Using equations listed in [24], we get

$$\lim_{k\to 1}\{K(k)[\pm Z(\varphi,k)-\sin\varphi]\}=-\frac{1}{2}\ln\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \tag{5.19}$$

where plus or minus should be used for $\varphi<\pi/2$ or $\varphi>\pi/2$, respectively. Let us consider equation (5.13). Using equations (5.19) and (4.21) we see that singular terms containing $K(k)$

cancel, and we arrive at the following result:

$$\lim_{m_3\to 0} I_3 = \frac{\pi^2}{8p^2}\left\{\sqrt{Q_+}\left(p^2+m_1^2+m_2^2\right)+\frac{1}{2}\left(p^2-m_1^2-m_2^2\right)^2\ln\left(\frac{p^2-m_1^2-m_2^2+\sqrt{Q_+}}{p^2-m_1^2-m_2^2-\sqrt{Q_+}}\right)\right.$$
$$-\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)^2\ln\left(\frac{p^2+m_1^2-m_2^2+\sqrt{Q_+}}{p^2+m_1^2-m_2^2-\sqrt{Q_+}}\right)$$
$$\left.-\frac{1}{2}\left(p^2-m_1^2+m_2^2\right)^2\ln\left(\frac{p^2-m_1^2+m_2^2+\sqrt{Q_+}}{p^2-m_1^2+m_2^2-\sqrt{Q_+}}\right)\right\} \qquad (5.20)$$

where $Q_+ = \lambda\left(p^2, m_1^2, m_2^2\right)$ in this limit. It is easy to check that this expression is equivalent to known results (see, e.g., [3, 8]). The advantage of our approach is that the symmetry with respect to any of the remaining masses is always explicit, whereas non-symmetric expressions such as equation (1.7) lead to the answers which are not explicitly symmetric (cf equation (57) of [8]).

6. Conclusion

We have considered several representations for the three-particle phase space, exploring their symmetry properties and geometrical meaning. It was shown that the angles φ_i defined in equations (4.20) and (4.21) are convenient to describe the results for the three-particle phase-space integral I_3. In terms of the Jacobian Z function (related to the elliptic integral Π through equation (5.14)), the result for I_3 in four dimensions is given in equation (5.13). It is very compact and explicitly symmetric with respect to all masses m_i. Note that the three zeta functions $Z(\varphi_i, k)$ are connected through the relation (5.10). This relation can be obtained by comparing the representation (5.7) for two-dimensional integral $I_3^{(2)}$ with another representation obtained by using the delta function properties.

In this way, we have shown how to transcribe the unsymmetric evaluation (1.2) of the phase-space integral into a form which is manifestly symmetric in the masses of the three decay products. Of course, the practical importance of this exercise is rather restricted, since (1.2) can be worked out numerically anyhow. Nevertheless, our result has an elegant structure and theoretical significance as it bears upon properties of elliptic functions which arise from elimination of variables in equations (2.13) and (2.16).

We have also considered the six-dimensional case. The result for $I_3^{(6)}$ is given in equation (5.15), also expressed in terms of $Z(\varphi_i, k)$.

Acknowledgments

We are pleased to acknowledge financial support from the Australian Research Council under grant number A00000780. Partial support from the Deutsche Forschungsgemeinschaft (AD) is also acknowledged.

Appendix. Elliptic integrals

The normal elliptic integrals of the first and second kind are defined as

$$F(\varphi, k) = \int_0^{\sin\varphi} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} \qquad (A.1)$$

$$E(\varphi, k) = \int_0^{\sin\varphi} \mathrm{d}t \sqrt{\frac{1-k^2t^2}{1-t^2}} = \int_0^{\varphi} \mathrm{d}\psi \sqrt{1-k^2\sin^2\psi}. \tag{A.2}$$

At $\varphi = \pi/2$ we get the complete elliptic integrals,

$$K(k) = F\left(\tfrac{\pi}{2}, k\right) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2}\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\middle| k^2\right) \tag{A.3}$$

$$E(k) = E\left(\tfrac{\pi}{2}, k\right) = \int_0^1 \mathrm{d}t \sqrt{\frac{1-k^2t^2}{1-t^2}} = \frac{\pi}{2}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}\\ 1\end{matrix}\middle| k^2\right) \tag{A.4}$$

$$\Pi(c, k) = \int_0^1 \frac{\mathrm{d}t}{(1-ct^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2} F_1\left(\tfrac{1}{2}; 1, \tfrac{1}{2}; 1 | c, k^2\right) \tag{A.5}$$

where F_1 is the Appell hypergeometric function of two arguments.

The Jacobian zeta function, $Z(\beta, k)$, is defined through

$$K(k)Z(\beta, k) = K(k)E(\beta, k) - E(k)F(\beta, k). \tag{A.6}$$

We will assume that $0 \leqslant k < 1$. (In the limit $k \to 1$, $F(\beta, k)$ and $K(k)$ are singular.) From the definition (A.6) it is obvious that $Z\left(\frac{\pi}{2}, k\right) = 0$. Moreover, using symmetry properties of $E(\beta, k)$ and $F(\beta, k)$ (see in [24]),

$$E\left(\tfrac{\pi}{2} + \delta, k\right) = 2E(k) - E\left(\tfrac{\pi}{2} - \delta, k\right) \qquad F\left(\tfrac{\pi}{2} + \delta, k\right) = 2K(k) - F\left(\tfrac{\pi}{2} - \delta, k\right)$$

we get

$$Z\left(\tfrac{\pi}{2} + \delta, k\right) = -Z\left(\tfrac{\pi}{2} - \delta, k\right). \tag{A.7}$$

To represent the elliptic functions $\Pi(\alpha_i, k)$ occurring in equation (1.7) in terms of Z functions, we can use

$$\Pi\left(\alpha_i^2, k\right) = K(k) + \frac{\alpha_i K(k) Z(\beta_i, k)}{\sqrt{\left(1-\alpha_i^2\right)\left(k^2-\alpha_i^2\right)}} \tag{A.8}$$

with $\beta_i = \arcsin(\alpha_i/k)$. Equation (A.8) corresponds to case III on p 229 of [24], when $0 \leqslant \alpha_i^2 < k^2$.

The following addition formulae from [24] (p 34, equation (142.01)) are needed:

$$Z(\beta_1, k) \pm Z(\beta_2, k) = Z(\varphi_\pm, k) \pm k^2 \sin\beta_1 \sin\beta_2 \sin\varphi_\pm \tag{A.9}$$

where the angles

$$\varphi_\pm = 2\arctan\left[\frac{\sin\beta_1\sqrt{1-k^2\sin^2\beta_2} \pm \sin\beta_2\sqrt{1-k^2\sin^2\beta_1}}{\cos\beta_1 + \cos\beta_2}\right]. \tag{A.10}$$

In fact, the angles φ_- and φ_+ correspond to the angles φ_1 and φ_2 (see equations (4.20) and (4.21)), respectively. Moreover, using the same addition formula (A.9) with $\beta_{1,2}$ substituted by $\varphi_{1,2}$, we get the symmetric connection (5.10) between $Z(\varphi_i, k)$ $(i = 1, 2, 3)$.

To derive the result given in equation (5.12), one can use the integral tables of [24], along with the following relation:

$$2Z(\beta_1, k) = -Z(\varphi_3, k) + \frac{2k^2\sin^3\beta_1\cos\beta_1\sqrt{1-k^2\sin^2\beta_1}}{1-k^2\sin^4\beta_1}. \tag{A.11}$$

It corresponds to the last two lines of equation (141.01) on p 33 of [24], where $\varphi \leftrightarrow \pi - \varphi_3$.

References

[1] Dalitz R H 1953 *Phil. Mag.* **44** 1068
[2] Fabri E 1954 *Nuovo Cimento* **11** 479
[3] Almgren B 1968 *Ark. Fys.* **38** 161
[4] Bashir A, Delbourgo R and Roberts M L 2001 *J. Math. Phys.* **42** 5553
[5] Hagedorn R 1962 *Nuovo Cimento* **25** 1017
[6] Delbourgo R and Roberts M L 2003 *J. Phys. A: Math. Gen.* **36** 1719
[7] Källen G 1964 *Elementary Particle Physics* (Reading, MA: Addison-Wesley)
[8] Bauberger S, Berends F A, Böhm M and Buza M 1995 *Nucl. Phys.* B **434** 383
[9] Tarasov O V 1997 *Nucl. Phys.* B **502** 455
[10] Davydychev A I and Smirnov V A 1999 *Nucl. Phys.* B **554** 391
[11] Aste A and Trautmann D 2003 *Can. J. Phys.* **81** 1433
[12] Mandelstam S 1955 *Proc. R. Soc. Lond.* A **233** 248
Mandelstam S 1959 *Phys. Rev.* **115** 1741
[13] Kibble T W B 1960 *Phys. Rev.* **117** 1159
[14] Arbuzov B A, Boos E E, Kurennoy S S and Turashvili K Sh 1986 *Yad. Fiz.* **44** 1565
Arbuzov B A, Boos E E, Kurennoy S S and Turashvili K Sh 1986 *Sov. J. Nucl. Phys.* **44** 1017 (Engl. Transl.)
[15] Mendels E 1978 *Nuovo Cimento* A **45** 87
Mendels E 2002 *J. Math. Phys.* **43** 3011
[16] Prudnikov A P, Brychkov Yu A and Marichev O I 1990 *Integrals and Series* vol 2 (New York: Gordon and Breach)
[17] Davydychev A I 2000 *Phys. Rev.* D **61** 087701
[18] Byckling E and Kajantie K 1973 *Particle Kinematics* (New York: Wiley)
[19] Davydychev A I and Delbourgo R 1998 *J. Math. Phys.* **39** 4299
[20] Berends F A, Davydychev A I and Ussyukina N I 1998 *Phys. Lett.* B **426** 95
[21] Rajantie A K 1996 *Nucl. Phys.* B **480** 729
Rajantie A K 1998 *Nucl. Phys.* B **513** 761 (erratum)
[22] Groote S, Körner J G and Pivovarov A A 1999 *Nucl. Phys.* B **542** 515
Groote S and Pivovarov A A 2000 *Nucl. Phys.* B **580** 459
[23] Roberts M L 2003 *PhD Thesis* University of Tasmania, Hobart
[24] Byrd P F and Friedman M D 1954 *Handbook of Elliptic Integrals for Engineers and Physicists* (Berlin: Springer)
[25] Davydychev A I and Delbourgo R 2002 *Proc. 15th Biennial Congress of the Australian Institute of Physics (Sydney, Australia, July 2002)* (Adelaide: Causal Productions) p 144 (*Preprint* hep-th/0209233)